Victoria Junior College

2010 Preliminary Examinations
Mathematics H2 (9740) Paper 2

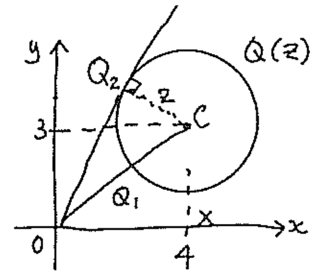
Section A

1

$|z-4-3i|=2$

$|z-(4+3i)|=2$

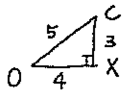
circle centred at $(4,3)$ with radius 2



(i) Least $|z|$ = distance OQ_1

$= OC - CQ_1$

$= 5-2=3$



$\therefore \quad a=3$

(ii) p is represented by point Q_1.

$\arg(p) = \angle XOQ_1 = \angle XOC$

$= \tan^{-1}\dfrac{3}{4}$

(iii) $\arg\left(\dfrac{z}{p}\right) = \arg(z) - \arg(p)$

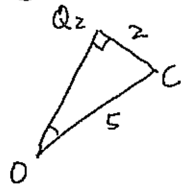
$= \angle XOQ - \angle XOC$

greatest $\arg\left(\dfrac{z}{p}\right) = \angle COQ_2$

$= \sin^{-1}\dfrac{2}{5}$

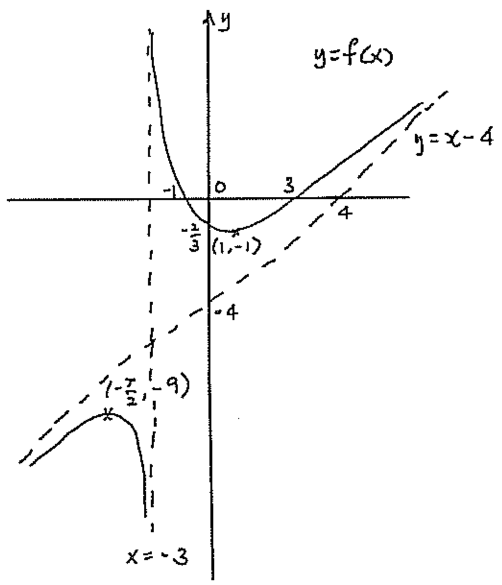
$= 0.41152$

≈ 0.41 (2dp)



2

(i)



(ii)

$y=[f(x)]^2$

$\dfrac{dy}{dx} = 2f(x)f'(x)$

$\dfrac{dy}{dx}=0 \Rightarrow f(x)=0$ or $f'(x)=0$

$\therefore$ x-coords of stat pts are $-1, 3, -\frac{7}{2}$ and 1.

3(a)

(i) $\Pi: \underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=4$

$\Pi_1: z=0 \Rightarrow \underset{\sim}{r}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=0$

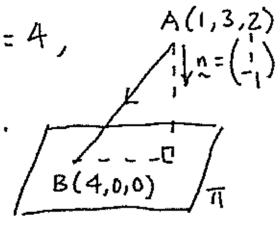
$\cos\theta = \dfrac{\begin{pmatrix}1\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}{\sqrt{1^2+1^2+1^2}\sqrt{1^2}} = \dfrac{-1}{\sqrt{3}}$

$\theta = 125.264°$

$\therefore$ acute $\angle$ betw the two planes is $180° - 125.264° = 54.736° \approx 54.7°$

(ii) Since $\begin{pmatrix}4\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=4$,

$B(4,0,0)$ lies in Π.



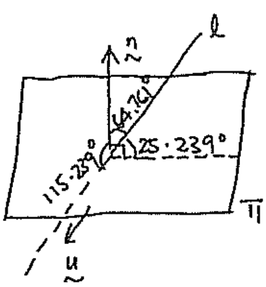
$\perp$ distance

$= |\overrightarrow{AB}\cdot\hat{\underset{\sim}{n}}|$

$= \left|\begin{pmatrix}3\\-3\\-2\end{pmatrix}\cdot\dfrac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix}\right| = \dfrac{2}{\sqrt{3}}$ or $\dfrac{2\sqrt{3}}{3}$

(b) $\Pi: \underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\-3\end{pmatrix}=4$

$l: \underset{\sim}{r} = \begin{pmatrix}1\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}, \lambda\in\mathbb{R}$



$\cos\theta = \dfrac{\begin{pmatrix}1\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}}{\sqrt{1^2+1^2+3^2}\sqrt{1^2+1^2}} = \dfrac{-2}{\sqrt{11}\sqrt{2}}$

$\theta = 115.239°$

acute $\angle$ betw l and $\begin{pmatrix}1\\3\end{pmatrix}$ is

$180° - 115.239° = 64.761°$

acute $\angle$ betw l and π is

$90° - 64.761° = 25.239° \approx 25.2°$

4(i)

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$$

$$= 1 + \left(-\tfrac{1}{2}\right)(-x^2) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-x^2)^2 + \ldots$$

$$= 1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \ldots$$

(ii)

Since $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$

$$\sin^{-1}x + c \approx \int 1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \ldots\,dx$$

$$= x + \tfrac{1}{6}x^3 + \ldots$$

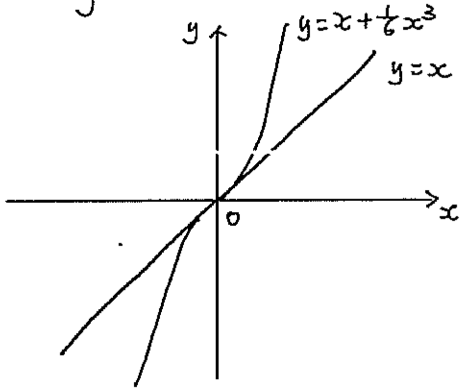
When $x=0$, $c=0$

$\therefore \sin^{-1}x \approx x + \frac{1}{6}x^3$

(iii) $y = \sin^{-1}x \approx x + \frac{1}{6}x^3$

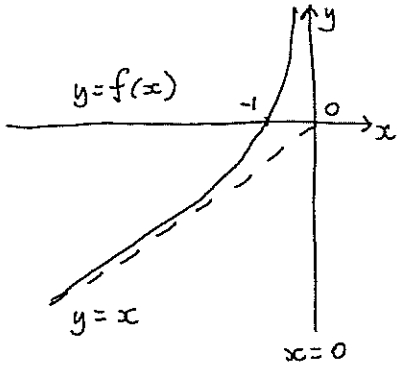
At $(0,0)$. $\frac{dy}{dx} = 1$

So eq of tangent at the origin is $y = x$



5(i)

$f: x \mapsto x - \frac{1}{x}$, $x<0$



(ii) Let $y = f(x)$

$$y = x - \frac{1}{x}$$

$$xy = x^2 - 1$$

$$x^2 - yx - 1 = 0$$

$$\left(x - \frac{y}{2}\right)^2 - \frac{y^2}{4} - 1 = 0$$

$$\left(x - \frac{y}{2}\right)^2 = \frac{y^2}{4} + 1$$

$$\left|x - \frac{y}{2}\right| = \sqrt{\frac{y^2}{4}+1} \quad \text{or} \quad x - \frac{y}{2} = \pm\sqrt{\frac{y^2}{4}+1}$$

Since $x<0$, $x - \frac{y}{2} = -\sqrt{\frac{y^2}{4}+1}$

$$\therefore x = \frac{y}{2} - \frac{1}{2}\sqrt{y^2+4}$$

$f^{-1}: x \mapsto \frac{x}{2} - \frac{1}{2}\sqrt{x^2+4}$, $x \in \mathbb{R}$

(iii) $f^{-1}(x) < -4$

Since f is increasing,

$$f f^{-1}(x) < f(-4)$$

$$x < -4 - \frac{1}{(-4)}$$

$$\therefore x < -\frac{15}{4}$$

(iv) $g: x \mapsto -x^2$, $x \in \mathbb{R}$

$R_g = (-\infty, 0]$, $D_f = (-\infty, 0)$

Since $R_g \nsubseteq D_f$, fg does not exist

(v) $h: x \mapsto \frac{2010}{x}$, $x \in \mathbb{R}, x>0$

$$h^2(x) = 2010 / \left(\frac{2010}{x}\right) = x$$

$$h^{21}(x) = h(h^{20}(x))$$

$$= h(x) = \frac{2010}{x}$$

Section B

6 P(winning \$3)

$= P(\text{spinning 1 three times})$

$= \frac{1}{n} \times \frac{1}{n} \times \frac{1}{n} = \frac{1}{n^3}$ (shown)

(i) P(winning nothing)

$= P(\text{spinning "skunk" in one of the three rounds})$

$= \frac{1}{n} + \frac{n-1}{n} \times \frac{1}{n} + \left(\frac{n-1}{n}\right)^2 \frac{1}{n}$ OR $1 - \left(\frac{n-1}{n}\right)^3$

(ii) P(win nothing | win \$3 in 1st round)

$= P(\text{spinning "skunk" in 2nd or 3rd round})$

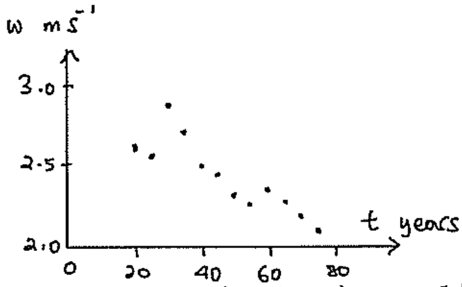
$= \frac{1}{n} + \frac{n-1}{n} \times \frac{1}{n}$ OR $1 - \left(\frac{n-1}{n}\right)^2$

(Alt) $\text{Prob} = \frac{P(\text{win \$3 in 1st round AND nothing at the end})}{P(\text{win \$3 in 1st round})}$

$= \frac{P(\text{3, skunk OR 3, non-skunk, skunk})}{P(\text{3 in 1st round})}$

$= \frac{\frac{1}{n} \times \frac{1}{n} + \frac{1}{n} \times \frac{n-1}{n} \times \frac{1}{n}}{\frac{1}{n}} = \frac{1}{n} + \frac{n-1}{n} \times \frac{1}{n}$

7(i)



(ii) w on t, because t is the independent variable.

(iii) The regression line is suitable since the points on the scatter diagram, for $30 \le t \le 55$, lie close to a straight line and the absolute value of the correlation coefficient is close to 1.

(iv)(a) With w and $\frac{1}{t}$,

$r = 0.99328 \approx 0.993$

Since the absolute value of this correlation coefficient is closer to 1 than the previous value of -0.965, the linear model for w and $\frac{1}{t}$ is a better model compared to the linear model for w and t.

(b) Eq of reg line of w on $\frac{1}{t}$ is

$w = 1.5763 + 37.419 \frac{1}{t}$

$\therefore w = 1.58 + 37.4 \frac{1}{t}$

(c) When $t = 43$, $w = 2.45$.

Hence the max walking speed at the age of 43 years is 2.45 ms^{-1}.

8 Let X g be the mass of a Perayaan ball.

$E(X) = 420$, $Var(X) = 6^2$

$T = X_1 + \dots + X_{50}$

Since n is large, by CLT,

$T \sim N(21000, 1800)$ approx

$P(T > 20900) = 0.99079 \approx 0.991$

No. The sum of the masses of the 50 balls will be approximately normally distributed by Central Limit Theorem since the sample size is large.

Let Y g be the mass of a Jubalani ball.

$Y \sim N(440, 1^2)$

$E(50Y - T) = 50 \times 440 - 21000 = 1000$

$Var(50Y - T) = 50^2 \times 1^2 + 1800 = 4300$

$\therefore 50Y - T \sim N(1000, 4300)$

$P(|50Y - T| < 900)$

$= P(-900 < 50Y - T < 900)$

$= 0.063631 \approx 0.0636$

Assume that the masses of all 51 Perayaan and Jubalani balls are mutually independent.

9 Let X be the number of arrivals in a 10-min interval. $X \sim Po(10\lambda)$

With $\lambda = 0.8$, $X \sim Po(8)$

$P(X \le 10 \mid X > 5) = \frac{P(5 < X \le 10)}{P(X > 5)}$

$= \frac{P(X \le 10) - P(X \le 5)}{1 - P(X \le 5)} = \frac{0.81589 - 0.19124}{1 - 0.19124}$

$= 0.77235 \approx 0.772$

Let Y be the number of arrivals in the peak period of 30min. $Y \sim Po(30\lambda)$

$Y \sim N(30\lambda, 30\lambda)$ approx

$P(Y > 30) < 0.05 \Rightarrow 1 - P(Y \le 30) < 0.05$

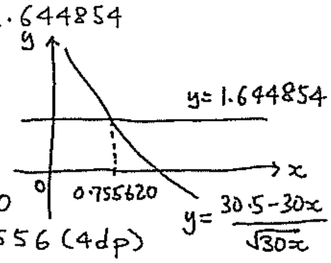
$\Rightarrow P(Y \le 30) > 0.95$

$P\left(Z \le \frac{30.5 - 30\lambda}{\sqrt{30\lambda}}\right) > 0.95$

From GC, $P(Z \le 1.644854) = 0.95$

$\therefore \frac{30.5 - 30\lambda}{\sqrt{30\lambda}} > 1.644854$

From GC,



$0 < \lambda < 0.755620$

$\therefore$ largest $\lambda = 0.7556$ (4dp)

10 A N B E L

A N

(a)(i) N _ _ E

Cases	No. of ways
N X X E	${}^1C_1 \times \frac{2!}{2!} = 1$
N X Y E	${}^4C_2 \times 2! = 12$

Total no. of ways = 1 + 12 = 13

(ii)

Cases	No. of ways
2 same, 2 same XXYY	${}^2C_2 \times \frac{4!}{2!2!} = 6$
2 same, 2 diff XXYZ	${}^2C_1 \times {}^4C_2 \times \frac{4!}{2!} = 144$
all diff XYZW	${}^5C_4 \times 4! = 120$

$\therefore$ Total no. of ways = 6 + 144 + 120 = 270

(b) P(tiles spell ANNA)

= P(A, A, N, N is picked and arranged as ANNA)

$= \frac{2}{7} \times \frac{2}{6} \times \frac{1}{5} \times \frac{1}{4} = \frac{1}{210}$

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11 Let X and Y dollars be the weekly earnings of casino 1 and 2 respectively.

$X \sim N(600\,000, 50\,000^2)$

$Y \sim N(700\,000, 75\,000^2)$

(i) $Y_1 + Y_2 \sim N(1\,400\,000, 1.125 \times 10^{10})$

$P(Y_1 + Y_2 > 1\,500\,000) = 0.17289 \approx 0.173$

(ii) $P(X > 650\,000) = 0.15866$

Let W be the no. of weeks, in a 12-week period, the weekly earnings exceeds \$650 000. $W \sim B(12, 0.15866)$

$P(W \geq 3) = 1 - P(W \leq 2)$

$= 1 - 0.70573 = 0.29427 \approx 0.294$

(iii) Let T dollars be the weekly tax on both casinos.

$T = 0.07X + 0.1Y$

$E(T) = 0.07(600\,000) + 0.1(700\,000) = 112\,000$

$Var(T) = 0.07^2(50\,000^2) + 0.1^2(75\,000^2) = 68\,500\,000$

$\therefore T \sim N(112\,000, 68\,500\,000)$

$P(T > 99\,000) = 0.94188 \approx 0.942$

Let S be the no. of weeks, out of 52, in which the weekly tax exceeds \$99 000

$S \sim B(52, 0.94188)$

Let S′ be the no. of weeks, out of 52, in which the weekly tax does not exceed \$99000.

$S' \sim B(52, 0.058124)$

Since n = 52 is large,

$np = 3.0225 < 5$,

$S' \sim P_0(3.0225)$ approx

$P(S \geq 45) = P(52 - S' \geq 45)$

$= P(S' \leq 7) = 0.98760 \approx 0.988$

12(a) No, the sampling is not random because the members are not equally likely to respond to their email.

(Alt) The probability of getting a response from any 226 out of 1000 members is not a constant.

(b) Let $\$\mu$ be the actual mean amount a customer spent on a Gundam model kit.

$H_0 : \mu = 40$

$H_1 : \mu < 40$

Level of significance : 5%

Test statistic : $T = \dfrac{\bar{X}-40}{\frac{S}{\sqrt{n}}}$

$n = 100$, $\sum x = 3500$, $\sum x^2 = 220400$

$\bar{x} = \dfrac{3500}{100} = 35$

$s^2 = \dfrac{1}{99}\left[220400 - \dfrac{3500^2}{100}\right] = \dfrac{97900}{99}$

$= 988.89$

Using t-test with $\nu = 99$,

p-value $= 0.057512$

Since p-value $= 0.0575 > 0.05$, H_0 is not rejected at the 5% significance level. Hence there is insufficient evidence that the distributor has overstated his claim.

We need to assume that the amount a customer spent on Gandam model kits is normally distributed, since the population variance is not known and a t-test is used.

(c) For a 5% significance level t-test with $\nu = 79$, the rejection region is $|T| > 1.9905$

For a 5% significance level z-test with $n = 80$, the rejection region is $|Z| > 1.9600$

Since both t and z are calculated from $\dfrac{\bar{x}-\mu_0}{\frac{s}{\sqrt{n}}}$, and the t-test concluded that H_0 is rejected, i.e.

$|z| = |t| = \left|\dfrac{\bar{x}-\mu_0}{s/\sqrt{n}}\right| > 1.9905 > 1.9600$,

the z-test will also reject H_0.

Hence (I) is correct.